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Black Rings

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Dedicated to the memory of Andrew Chamblin

Abstract

A black ring is a five-dimensional black hole with an event horizon of topology $S^1 \times S^2$. We provide an introduction to the description of black rings in general relativity and string theory. Novel aspects of the presentation include a new approach to constructing black ring coordinates and a critical review of black ring microscopics.

Contents

1 Introduction

The classical theory of black holes developed in the 1960's-70's produced a proof of the simplicity of four-dimensional black holes. An asymptotically flat, stationary black hole solution[1] of four-dimensional Einstein-Maxwell theory is fully specified by a handful of parameters—the black hole has 'no hair'. More strongly, the uniqueness theorems assert that these parameters are precisely those that correspond to conserved charges, namely, the mass M and angular momentum J, and possibly the charges Q associated to local gauge symmetries. Hence, the only black hole solution of the four-dimensional Einstein-Maxwell theory is the Kerr-Newman black hole. This result precludes the possibility that a black hole possesses higher multipole moments (*e.g.,* a mass quadrupole or a charge dipole) that are not completely fixed by the values of the conserved charges. Another consequence is that the microscopic states that are responsible for the large degeneracy implied by the Bekenstein-Hawking entropy, are invisible at the level of the classical gravitational theory.

We will review here the recent discovery that five-dimensional black holes exhibit qualitatively new properties not shared by their four-dimensional siblings. In short, non-spherical horizon topologies are possible, and conventional notions of black hole uniqueness do not apply [1]. This, and other recent developments —notably, the study of the interplay between black holes and black strings in theories with compact dimensions [2]— have made plainly clear that the physics of higher-dimensional black holes is largely *terra incognita*, and have prompted its exploration in earnest.

Although the higher-dimensional version of the Schwarzschild solution was found long ago [3], it was not until 1986, with the impetus provided by the development of string theory, that the higher-dimensional version of the Kerr solution was constructed by Myers and Perry (MP) [4]. Given that the Kerr black hole solution is unique in four dimensions, it may have seemed natural to expect black hole uniqueness also in higher dimensions.

We know that, at least in five dimensions, and very likely in $D \geq 5$ dimensions, this is not the case. A heuristic argument that suggests the possibility of black holes of non-spherical topology is the following. Take a neutral black string in five dimensions, constructed as the direct product of the Schwarzschild solution and a line, so the geometry of the horizon is $\mathbf{R} \times S^2$. Imagine bending this string to form a circle, so the topology is now $S^1 \times S^2$. In principle this circular string tends to contract, decreasing the radius of the S^1, due to its tension and gravitational self-attraction. However, we can make the string rotate along the S^1 and balance these forces against the centrifugal repulsion. Then we end up with a neutral rotating *black ring*: a black hole with an event horizon of topology $S^1 \times S^2$. Ref. [1] obtained an explicit solution of five-dimensional vacuum general relativity describing such an object.

[1]Asymptotic flatness and stationarity will be assumed of all solutions considered in this paper.

This was not only an example of non-spherical horizon topology, but it also turned out to be a counterexample to black hole uniqueness.

The discovery that black hole uniqueness is violated in higher dimensions was greeted with surprise but, with the benefit of hindsight, what we should really regard as surprising is that black hole uniqueness *is* valid in four dimensions. The no-hair property of four-dimensional black holes was regarded as very surprising at the time of its discovery, and the realization that it does not extend to higher dimensions serves to emphasize this.

The Forging of the Ring

Originally, an investigation of *static* solutions [5] combined with educated guesswork (which gave the solutions in [6, 7, 8] from which black rings are obtained via analytic continuation) was the way to the rotating black ring solutions of [1, 9]. Recently, these solutions have been rederived in a systematic manner via solution-generating techniques [10, 11]. The same techniques have also given vacuum black rings with rotation on the S^2 but without rotation along the S^1 ring circle [12] —however, an educated guess has independently given the same solution in a much more manageable form [13]. In contrast, the charged supersymmetric and non-supersymmetric black rings have, from the start, been constructed in a more systematic way.

A note on nomenclature

A black ring is defined to be a D-dimensional black hole for which the topology of (a spatial cross-section of) the event horizon is $S^1 \times S^{D-3}$. (So far, such solutions are only known in $D = 5$ but we allow for the possibility that such objects may exist for $D > 5$.) Often one wishes to distinguish black rings from topologically spherical black holes, which are often abbreviated to simply "black holes". Black rings are also black holes, but the context should eliminate any possible confusion.

Outline of this review

A main difficulty in understanding the black ring solutions appears to be the somewhat unfamiliar coordinate system in which they take their simplest known form. Therefore, we devote the next section 2 to a derivation and explanation of these coordinates. This may also be useful for obtaining adapted coordinates in other settings. Section 3 studies the neutral black ring and how it gives rise to non-uniqueness in five dimensions. Section 4 introduces black rings as charged sources of gauge fields. The particularly interesting case of supersymmetric black rings is analyzed in section 5. Section 6 examines the ideas underlying the microscopic description of black rings in string and M theory. In section 7 we discuss other developments related to the role of black rings in string/M theory. The final section briefly addresses some open issues and ideas for future work.

2 Ring coordinates

The rotation group in four spatial dimensions, $SO(4)$, contains two mutually commuting $U(1)$ subgroups, meaning that it is possible to have rotation in two independent rotation planes. In order to see this point more clearly, consider four-dimensional flat space and group the four spatial coordinates in two pairs, choosing polar coordinates for each of the two planes,

$$x^1 = r_1 \cos\phi, \quad x^2 = r_1 \sin\phi, \quad x^3 = r_2 \cos\psi, \quad x^4 = r_2 \sin\psi\,. \tag{1}$$

Rotations along ψ and ϕ generate two independent angular momenta J_ψ and J_ϕ. We will describe rings extending along the (x^3, x^4)-plane, and rotating along ψ, thus giving rise to non-vanishing J_ψ.

As is often the case in General Relativity, it is very convenient to work in adapted coordinates. A general idea to find them is to begin by constructing a foliation of flat space in terms of the equipotential surfaces of the field created by a source resembling the black hole one is seeking[2]. It turns out that, instead of considering the equipotential surfaces of a *scalar* field sourced by a ring, it is more convenient to work with the equipotential surfaces of a *2-form potential* $B_{\mu\nu}$. Thus we regard the ring as a circular string that acts as an electric source of the 3-form field strength $H = dB$, which satisfies the field equation

$$\partial_\mu(\sqrt{-g}H^{\mu\nu\rho}) = 0 \tag{2}$$

outside the ring source. Let us write four-dimensional flat space in the coordinates of (1)

$$d\mathbf{x}_4^2 = dr_1^2 + r_1^2 d\phi^2 + dr_2^2 + r_2^2 d\psi^2\,. \tag{3}$$

It is easy to construct the solution of (2) for a circular electric source at $r_1 = 0$, $r_2 = R$ and $0 \le \psi < 2\pi$ using methods familiar in classical electrodynamics, as (see [15])

$$\begin{aligned} B_{t\psi} &= \frac{R}{2\pi}\int_0^{2\pi} d\psi \frac{r_2\cos\psi}{r_1^2 + r_2^2 + R^2 - 2Rr_2\cos\psi} \\ &= -\frac{1}{2}\left(1 - \frac{R^2 + r_1^2 + r_2^2}{\Sigma}\right) \end{aligned} \tag{4}$$

where

$$\Sigma = \sqrt{(r_1^2 + r_2^2 + R^2)^2 - 4R^2 r_2^2}\,. \tag{5}$$

We can as easily find the electric-magnetic (Hodge) dual of this field. In five spacetime dimensions, $*H = F = dA$ where A is a one-form potential, so the dual of an electric string

[2] In [14] a similar approach is followed to obtain coordinates suitable for black holes and black strings on a Kaluza-Klein circle.

is a magnetic monopole—in this case a circular distribution of monopoles. Note that surfaces of constant A_ϕ will be orthogonal to surfaces of constant $B_{t\psi}$. For the dual of the field (4) one finds

$$A_\phi = -\frac{1}{2}\left(1+\frac{R^2-r_1^2-r_2^2}{\Sigma}\right) . \tag{6}$$

Now define coordinates y and x that correspond to constant values of $B_{t\psi}$ and A_ϕ, respectively. A convenient choice is

$$y = -\frac{R^2+r_1^2+r_2^2}{\Sigma}\,, \qquad x = \frac{R^2-r_1^2-r_2^2}{\Sigma}\,, \tag{7}$$

with inverse

$$r_1 = R\frac{\sqrt{1-x^2}}{x-y}\,, \qquad r_2 = R\frac{\sqrt{y^2-1}}{x-y}\,. \tag{8}$$

Observe that the coordinate ranges are

$$-\infty \le y \le -1\,, \qquad -1 \le x \le 1 \tag{9}$$

with $y=-\infty$ corresponding to the location of the ring source, and asymptotic infinity recovered as $x \to y \to -1$. The axis of rotation around the ψ direction, $r_2 = 0$ (actually not a line but a plane) is at $y=-1$, and the axis of rotation around ϕ, $r_1 = 0$, is divided into two pieces: $x = 1$ is the disk $r_2 \le R$, and $x=-1$ is its complement outside the ring, $r_2 \ge R$. In these coordinates the flat metric (3) becomes

$$d\mathbf{x}_4^2 = \frac{R^2}{(x-y)^2}\left[(y^2-1)d\psi^2 + \frac{dy^2}{y^2-1} + \frac{dx^2}{1-x^2} + (1-x^2)d\phi^2\right] . \tag{10}$$

This is depicted in fig. 1, where we present a section at constant ψ and ϕ (as well as the antipodal sections at $\psi+\pi$, $\phi+\pi$ for greater clarity).

We can rewrite this same foliation of space in a manner that is particularly appropriate in the region near the ring. Define coordinates r and θ as

$$r = -\frac{R}{y}\,, \qquad \cos\theta = x\,, \tag{11}$$

with

$$0 \le r \le R\,, \qquad 0 \le \theta \le \pi\,. \tag{12}$$

The flat metric (10) becomes

$$d\mathbf{x}_4^2 = \frac{1}{\left(1+\frac{r\cos\theta}{R}\right)^2}\left[\left(1-\frac{r^2}{R^2}\right)R^2 d\psi^2 + \frac{dr^2}{1-r^2/R^2} + r^2\left(d\theta^2+\sin^2\theta\, d\phi^2\right)\right] . \tag{13}$$

Note that the apparent singularity[3] at $r=R$ actually corresponds to the ψ-axis of rotation.

[3]Which, not by accident, may remind some readers of the deSitter horizon.

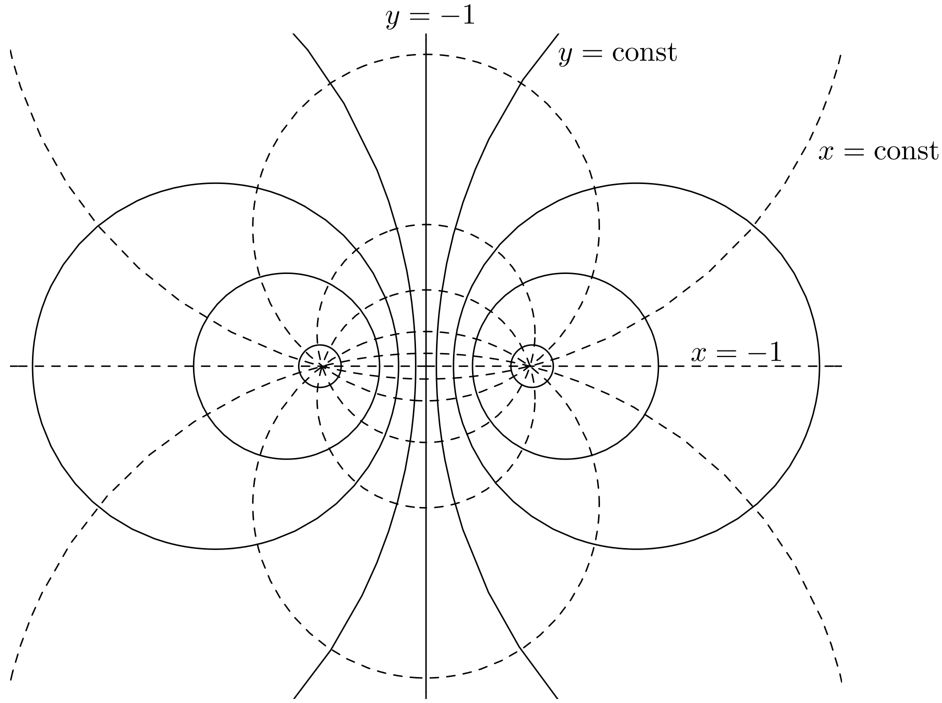

Figure 1: Ring coordinates for flat four-dimensional space, on a section at constant ϕ and ψ (and $\phi + \pi$, $\psi + \pi$). Dashed circles correspond to spheres at constant $|x| \in [0, 1]$, solid circles to spheres at constant $y \in [-\infty, -1]$. Spheres at constant y collapse to zero size at the location of the ring of radius R, $y = -\infty$. The disk bounded by the ring is an axis for ∂_ϕ at $x = +1$.

It is now manifest that the surfaces at constant r, *i.e.,* constant y, have ring-like topology $S^2 \times S^1$, where the S^2 is parametrized by (θ, ϕ) and the S^1 by ψ. The black rings will have their horizons (and ergosurfaces) at constant values of y, or r, and so their topology will be clear in these coordinates.

In the flat space (13), the S^2's at constant r are actually metrically round spheres, centered at $r_2 = R/\sqrt{1 - r^2/R^2}$, with radius $r/\sqrt{1 - r^2/R^2}$. Due to the overall prefactor in (13), this is not quite obvious. However, for small spheres $r \ll R$ the prefactor is ≈ 1, and r and θ recover their conventional interpretation as the radius and polar angle on the spheres[4]. Hence the coordinates r and θ are natural in the region of small r, but they look bizarre at larger distances. In particular asymptotic infinity corresponds to $r \cos\theta = -R$. The coordinates (x, y) are physically opaque, but they preserve a symmetry under exchange $x \leftrightarrow y$ that is otherwise obscured, and allow for more compact expressions.

Incidentally, Σ^{-1} solves the Laplace equation for a ring sourcing a scalar field, $\nabla^2 \Sigma^{-1} =$

[4]The surfaces at constant $|x|$ and constant ϕ are also round spheres, centered at $r_1 = R|x|/\sqrt{1 - x^2}$ and with radius $R/\sqrt{1 - x^2}$.

0. Since $\Sigma^{-1} = (x-y)/2R^2$, we see that surfaces of constant scalar potential do not correspond to constant x nor y, except in the limit of large negative y ($r \ll R$) where $\Sigma^{-1} \simeq -y/2R^2 = 1/2Rr$. It is possible to construct coordinates adapted to surfaces of constant Σ and their gradient surfaces, but the form of the black ring solutions becomes somewhat more complicated. The (x,y) coordinates, being adapted to the two-form potential B, also facilitate greatly the analysis of solutions with gauge dipoles, in particular of supersymmetric black rings.

3 Neutral Black Ring

3.1 Spacetime geometry

The metric for the black ring geometry preserves most of the basic structure of (10), but now it contains additional functions that encode the non-zero curvature produced by the black ring. In (x,y) coordinates these functions admit a particularly simple form, as they can be written as linear functions of x and y.

The solution has been given in three related forms in [1], [16, 17], and [9], the latter two forms being more convenient than the original one. The form given in [9] appears to be more fundamental, since black rings with a dipole, or with rotation in the S^2 [13], are more naturally connected to this version of the solution. The metric is[5]

$$\begin{aligned} ds^2 &= -\frac{F(y)}{F(x)}\left(dt - C\,R\,\frac{1+y}{F(y)}\,d\psi\right)^2 \\ &\quad + \frac{R^2}{(x-y)^2}\,F(x)\left[-\frac{G(y)}{F(y)}d\psi^2 - \frac{dy^2}{G(y)} + \frac{dx^2}{G(x)} + \frac{G(x)}{F(x)}d\phi^2\right], \end{aligned} \tag{14}$$

where

$$F(\xi) = 1 + \lambda\xi, \qquad G(\xi) = (1-\xi^2)(1+\nu\xi)\,, \tag{15}$$

and

$$C = \sqrt{\lambda(\lambda-\nu)\frac{1+\lambda}{1-\lambda}}\,. \tag{16}$$

The dimensionless parameters λ and ν must lie in the range

$$0 < \nu \leq \lambda < 1\,. \tag{17}$$

[5]If we denote quantities in [17] with a hat, then the relationship is $x = \frac{\hat{x}-\hat{\lambda}}{1-\hat{\lambda}\hat{x}}$, $y = \frac{\hat{y}-\hat{\lambda}}{1-\hat{\lambda}\hat{y}}$, $(\phi,\psi) = \frac{1-\hat{\lambda}\hat{\nu}}{\sqrt{1-\hat{\lambda}^2}}(\hat{\phi},\hat{\psi})$, $\nu = \frac{\hat{\lambda}-\hat{\nu}}{1-\hat{\lambda}\hat{\nu}}$, and $\lambda = \hat{\lambda}$.

When both λ and ν vanish we recover flat spacetime in the form (10). R sets the scale for the solution, and λ and ν are parameters that characterize the shape and rotation velocity of the ring, as we shall clarify presently.

Although we shall work primarily with the metric in these coordinates and parameters, since they allow for more compact expressions, it is instructive to also consider the (r, θ) coordinates introduced in (11), as well as to redefine the parameters $(\nu, \lambda) \to (r_0, \sigma)$ as

$$\nu = \frac{r_0}{R}, \qquad \lambda = \frac{r_0 \cosh^2 \sigma}{R}. \tag{18}$$

The solution becomes perhaps uglier,

$$\begin{aligned} ds^2 &= -\frac{\hat{f}}{\hat{g}} \left(dt - r_0 \sinh\sigma \cosh\sigma \sqrt{\frac{R + r_0 \cosh^2\sigma}{R - r_0 \cosh^2\sigma}} \, \frac{\frac{r}{R} - 1}{r \hat{f}} \, R \, d\psi \right)^2 \\ &\quad + \frac{\hat{g}}{\left(1 + \frac{r \cos\theta}{R}\right)^2} \left[\frac{f}{\hat{f}} \left(1 - \frac{r^2}{R^2}\right) R^2 d\psi^2 + \frac{dr^2}{(1 - \frac{r^2}{R^2}) f} + \frac{r^2}{g}\, d\theta^2 + \frac{g}{\hat{g}}\, r^2 \sin^2\theta \, d\phi^2 \right] \end{aligned} \tag{19}$$

where

$$f = 1 - \frac{r_0}{r}, \qquad \hat{f} = 1 - \frac{r_0 \cosh^2\sigma}{r}, \tag{20}$$

and

$$g = 1 + \frac{r_0}{R} \cos\theta, \qquad \hat{g} = 1 + \frac{r_0 \cosh^2\sigma}{R} \cos\theta. \tag{21}$$

Consider the limit

$$r,\ r_0,\ r_0 \cosh^2\sigma \ll R \tag{22}$$

in which $g,\ \hat{g} \approx 1$, and redefine $\psi = z/R$. Then (19) becomes exactly the metric for a boosted black string, extended along the direction z, and with boost parameter σ. The horizon is at $r = r_0$, and absence of conical singularities requires that ψ be identified with period 2π so the string is periodically identified with radius R: $z \sim z + 2\pi R$. Hence the limit (22) corresponds to taking the ring radius R much larger than the ring thickness r_0, and focusing on the region near the ring $r \sim r_0$.

This gives precise meaning to the heuristic construction of a black ring as a boosted black string bent into circular shape. It also allows to give an approximate interpretation to λ and ν. According to (18), the parameter ν measures the ratio between the radius of the S^2 at the horizon, r_0, and the radius of the ring R. So smaller values of ν correspond to thinner rings. Also, λ/ν is a measure of the speed of rotation of the ring. More precisely, $\sqrt{1 - (\nu/\lambda)}$ can be approximately identified with the local boost velocity $v = \tanh\sigma$.

We now turn to a general analysis of the metric in the form (14). The coordinates x and y vary within the same range as in (9) and are interpreted in essentially the same manner as we saw in the previous section. An important difference, though, is that for general values of the parameters in (17) the orbits of $\partial/\partial\psi$ and $\partial/\partial\phi$ do not close off smoothly at their respective axes, but in general have conical singularities there. To avoid them at $x=-1$ and $y=-1$ the angular variables must be identified with periodicity

$$\Delta\psi = \Delta\phi = 4\pi\frac{\sqrt{F(-1)}}{|G'(-1)|} = 2\pi\frac{\sqrt{1-\lambda}}{1-\nu}\,. \tag{23}$$

To avoid also a conical singularity at $x=+1$ we must have $\Delta\phi = 2\pi\sqrt{1+\lambda}/(1+\nu)$. This is compatible with (23) only if we take the two parameters λ, ν, to satisfy

$$\lambda = \frac{2\nu}{1+\nu^2}\,. \tag{24}$$

Fixing λ to this value leaves only two independent parameters in the solution, R and ν. In fact this is as expected on physical grounds: given, say, the mass and the radius of the ring, the angular momentum must be tuned so that the centrifugal force balances the tension and self-attraction of the ring, thus leaving only two free parameters. Demanding the absence of conical singularities, as in (24), actually corresponds to the condition that the system is balanced without any external forces.

Note that in terms of the boost parameter σ introduced in (18), and in the limit of thin rings (22), the equilibrium value from (24) becomes

$$|\sinh\sigma| \to 1 \tag{25}$$

or equivalently, the velocity $|v| \to 1/\sqrt{2}$. This happens to be the value of the boost that makes the ADM pressure of the black string vanish, $T_{zz}=0$ [17]. The latter holds for all known black rings in equilibrium, including dipole rings [9]. Presumably it applies in more generality, *e.g.,* for higher-dimensional black rings and black objects with more complicated horizon topology.

With the choices (23) and (24) for the parameters the solution becomes asymptotically flat as $x\to y\to -1$. Since the geometry is distorted by the presence of curvature, in order to go to manifestly asymptotically flat coordinates we have to modify (8) slightly. We set

$$\tilde r_1 = \tilde R\frac{\sqrt{2(1+x)}}{x-y}\,,\quad \tilde r_2 = \tilde R\frac{\sqrt{-2(1+y)}}{x-y}\,,\quad \tilde R^2 = R^2\frac{1-\lambda}{1-\nu}\,,\quad (\tilde\psi,\tilde\phi) = \frac{2\pi}{\Delta\psi}(\psi,\phi) \tag{26}$$

(note that we are taking x and y close to -1, and that $\tilde\psi,\tilde\phi$ have canonical periodicity). Then (14) asymptotes to the flat space metric (3), now with 'tilded' coordinates $\tilde r_{1,2},\tilde\psi,\tilde\phi$.

Note that $F(y)$ vanishes at $y=-1/\lambda$. Nevertheless, it is easy to check that the metric and its inverse are smooth there. This locus corresponds to a timelike surface in spacetime at which $\partial/\partial t$ changes from timelike to spacelike, *i.e.,* it is an *ergosurface.* A spatial cross section of this surface has topology $S^1 \times S^2$.

At $y = -1/\nu$ the metric becomes singular, but we can show that this is only a coordinate singularity by the transformation $(t, \psi) \to (v, \psi')$ as

$$dt = dv - CR\frac{1+y}{G(y)\sqrt{-F(y)}}dy\,, \qquad d\psi = d\psi' + \frac{\sqrt{-F(y)}}{G(y)}dy\,. \tag{27}$$

In these coordinates the metric is

$$\begin{aligned} ds^2 &= -\frac{F(y)}{F(x)}\left(dv - CR\frac{1+y}{F(y)}d\psi'\right)^2 \\ &\quad + \frac{R^2}{(x-y)^2}F(x)\left[-\frac{G(y)}{F(y)}d\psi'^2 + 2\frac{d\psi'\,dy}{\sqrt{-F(y)}} + \frac{dx^2}{G(x)} + \frac{G(x)}{F(x)}d\phi^2\right]\,, \end{aligned} \tag{28}$$

which is manifestly regular at $y = -1/\nu$. Let

$$V = \frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial\tilde{\psi}} = \frac{\partial}{\partial v} + \Omega\frac{\partial}{\partial\tilde{\psi}'}, \tag{29}$$

where $\tilde{\psi}' = (2\pi/\Delta\psi)\psi'$ and

$$\Omega = \frac{1}{R}\sqrt{\frac{\lambda-\nu}{\lambda(1+\lambda)}}\,. \tag{30}$$

Then V is null at $y = -1/\nu$ and $V_\mu dx^\mu$ is a positive multiple of dy, from which it follows that $y = -1/\nu$ is a Killing horizon with angular velocity Ω. In the limit (22) of a thin ring we recover $\Omega R = \tanh\sigma$.

This horizon has spatial topology $S^1 \times S^2$, although the S^2 is distorted away from perfect sphericity. At $y = -\infty$ the invariant $R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho}$ blows up, which corresponds to an inner spacelike singularity .

The Myers-Perry black hole with rotation in a single plane is contained within the family of solutions (14) as the particular limit[6] in which $R \to 0$, and $\lambda, \nu \to 1$, while maintaining fixed the parameters a, m,

$$m = \frac{2R^2}{1-\nu}\,, \qquad a^2 = 2R^2\frac{\lambda-\nu}{(1-\nu)^2}\,, \tag{31}$$

changing coordinates $(x, y) \to (r, \theta)$,

$$\begin{aligned} x &= -1 + 2\left(1 - \frac{a^2}{m}\right)\frac{R^2\cos^2\theta}{r^2 - (m-a^2)\cos^2\theta}\,, \\ y &= -1 - 2\left(1 - \frac{a^2}{m}\right)\frac{R^2\sin^2\theta}{r^2 - (m-a^2)\cos^2\theta}\,, \end{aligned} \tag{32}$$

[6]The limit is much less singular in the coordinates of [16, 17].

and rescaling $(\psi,\phi) \to \sqrt{\frac{m-a^2}{2R^2}}\,(\psi,\phi)$ so they now have canonical periodicity 2π. Then we recover the metric

$$ds^2 = -\left(1-\frac{m}{\Sigma}\right)\left(dt + \frac{ma\sin^2\theta}{\Sigma - m}\,d\psi\right)^2 + \Sigma\left(\frac{dr^2}{\Delta} + d\theta^2\right) + \frac{\Delta\sin^2\theta}{1-m/\Sigma}\,d\psi^2 + r^2\cos^2\theta\,d\phi^2\,, \tag{33}$$

$$\Delta \equiv r^2 - m + a^2\,, \qquad \Sigma \equiv r^2 + a^2\cos^2\theta \tag{34}$$

of the MP black hole rotating in the ψ direction. The extremal limit $m = a^2$ of the MP black hole actually corresponds to the same nakedly singular solution obtained as $\nu \to 1$ in (14).

3.2 Physical magnitudes and Non-Uniqueness

To demonstrate the absence of uniqueness for this family of solutions we need their two conserved charges: the mass and spin. These are obtained by examining the metric near asymptotic infinity, $x \to y \to -1$, in the more conventional coordinates of (26), and comparing to the linearized gravity analysis in [4].[7] We find

$$M = \frac{3\pi R^2}{4G}\frac{\lambda}{1-\nu}\,, \tag{35}$$

$$J = \frac{\pi R^3}{2G}\frac{\sqrt{\lambda(\lambda-\nu)(1+\lambda)}}{(1-\nu)^2}\,. \tag{36}$$

The horizon area and temperature (from the surface gravity $\kappa = 2\pi T$) are

$$\mathcal{A}_H = 8\pi^2 R^3 \frac{\nu^{3/2}\sqrt{\lambda(1-\lambda^2)}}{(1-\nu)^2(1+\nu)}\,, \tag{37}$$

$$T = \frac{1}{4\pi R}(1+\nu)\sqrt{\frac{1-\lambda}{\lambda\nu(1+\lambda)}}\,. \tag{38}$$

To analyze the physical properties of the solutions it is convenient to first fix the overall scale. Instead of fixing R, which has no invariant meaning, we shall fix the mass M. The solutions can then be characterized by dimensionless magnitudes obtained by dividing out an appropriate power of M or of GM (which has dimension (length)^2). So we define a dimensionless "reduced spin" variable j, conveniently normalized as

$$j^2 \equiv \frac{27\pi}{32G}\frac{J^2}{M^3}\,, \tag{39}$$

[7] Although note that the solutions of [4] are all rotating in a *negative* sense. Positive rotation corresponds to $g_{t\psi}$ negative near infinity.

(j^2 is often a more convenient variable than j), as well as a reduced area of the horizon,

$$a_H \equiv \frac{3}{16}\sqrt{\frac{3}{\pi}}\frac{\mathcal{A}_H}{(GM)^{3/2}}\,. \tag{40}$$

Above we argued that a black ring at equilibrium, *i.e.,* satisfying (24), has only one independent dimensionless parameter. Therefore at equilibrium the reduced area and spin, a_H and j, must be related. Using the results above, this can be expressed in parametric form as

$$a_H = 2\sqrt{\nu(1-\nu)}\,, \qquad j^2 = \frac{(1+\nu)^3}{8\nu} \qquad \text{(black ring)}\,, \tag{41}$$

with $0 < \nu \leq 1$.

For the spherical MP black hole (33) the corresponding relation can be found in [4], or also by taking the limit from the general ring solution as explained above in (31). The result is

$$a_H = 2\sqrt{2(1-j^2)} \qquad \text{(MP black hole)}\,. \tag{42}$$

The curves (41) and (42) are plotted and described in figure 2. The plot exhibits several unusual features. For instance, contrary to what happens for rotating black holes in four dimensions, and for the MP black hole in five dimensions, the angular momentum of the black ring (for fixed mass) is bounded below, but not above. But the most striking feature is that in the range $27/32 \leq j^2 < 1$ there exist one MP black hole and two black rings all with the same values of the mass and the spin. Since the latter are the only conserved quantities carried by these objects, we have an explicit violation of black hole uniqueness.

It is sometimes asserted that the existence of the black ring implies *per se* a violation of black hole uniqueness. However, we do not know of any a priori argument why the respective ranges of j for MP black holes and black rings should overlap[8]. Only through examination of the explicit solutions do we see that they are, respectively, $j^2 < 1$ and $j^2 \geq 27/32$, and so indeed they do overlap—but then only rather narrowly so! Observe also that it is not possible to recover a notion of uniqueness by fixing the horizon topology, since there can be two black rings with the same M and J.

4 Charges and Dipoles

For topologically spherical black holes (*e.g.,* the Kerr-Newman solution), the combination of electric charges and rotation gives rise to associated magnetic dipoles, which do not violate uniqueness since they do not provide parameters independent of the conserved charges.

[8]At least not an argument within classical relativity. But perhaps one might argue from thermodynamics that the phases in the diagram fig. 2 should exhibit the generic 'swallowtail' structure, as indeed they do.

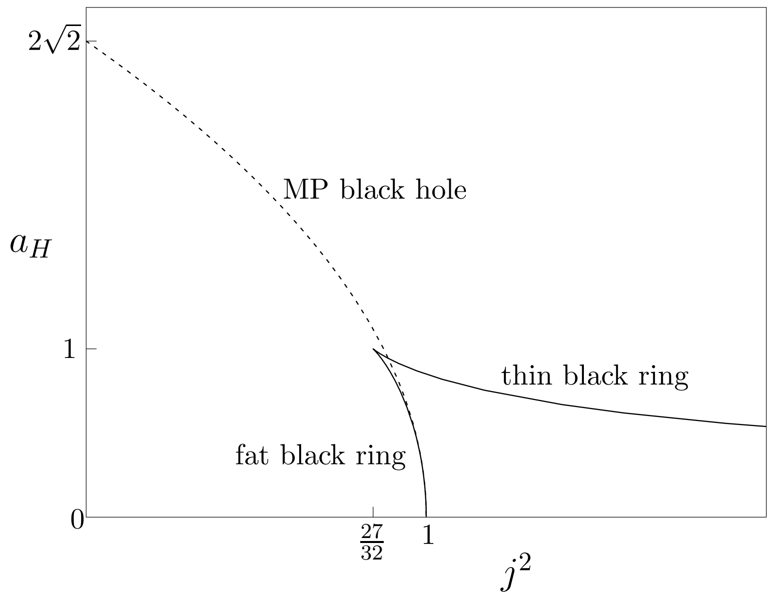

Figure 2: Horizon area a_H vs (spin)^2 j^2, for given mass, for the neutral rotating black ring (solid) and MP black hole (dotted). There are two branches of black rings, which branch off from the cusp at $(j^2, a_H) = (27/32, 1)$, and which are dubbed "thin" and "fat" according to their shape. When $27/32 < j^2 < 1$ we find the *Holey Trinity:* three different solutions —two black rings and one MP black hole— with the same dimensionless parameter j (*i.e.,* with the same mass and spin). The minimally spinning ring, with $j^2 = 27/32$, has a regular non-degenerate horizon, so it is not an extremal solution. Other interesting features are: At $j^2 = {a_H}^2 = 8/9$ the curves intersect and we find a MP black hole and a (thin) black ring both with the same mass, spin and area. The limiting solution at $(j^2, a_H) = (1, 0)$ is a naked singularity. Rapidly spinning black rings, $j^2 \to \infty$, become thinner and their area decreases as $a_H \sim 1/(j\sqrt{2})$.

Black rings can carry conserved gauge charges (the first example was obtained in [18]). More remarkably, they can also support gauge dipoles that are *independent* of all conserved charges, in fact they can be present even in the absence of any gauge charge. So, generically, these dipoles entail continuous violations of uniqueness. This is a much more drastic effect than the discrete, three-fold non-uniqueness that we have found for neutral rings.

The charges and dipoles of black rings actually provide the basis to interpret them as objects in string/M-theory. The five-dimensional supergravities of which these black rings are solutions are then most conveniently viewed as dimensional reductions of eleven-dimensional supergravity, the low-energy limit of M-theory, as we review next.

4.1 Dimensional Reduction to Five Dimensions

We start with eleven-dimensional supergravity, whose bosonic fields are the metric and a 3-form potential $\mathcal{A}$ with 4-form field strength $\mathcal{F} = d\mathcal{A}$. The action is

$$I_{11} = \frac{1}{16\pi G_{11}} \int \left(R_{11} \star_{11} 1 - \frac{1}{2}\mathcal{F} \wedge \star_{11}\mathcal{F} - \frac{1}{6}\mathcal{F} \wedge \mathcal{F} \wedge \mathcal{A} \right) , \tag{43}$$

where R_{11} and $\star_{11}$ denote the eleven-dimensional Ricci scalar and Hodge dual, respectively. We shall be interested in a five-dimensional supergravity theory obtained by dimensional reduction on T^6 using the Ansatz[9]

$$\begin{aligned} ds^2_{11} &= ds^2_5 + X^1 \left(dz_1^2 + dz_2^2\right) + X^2 \left(dz_3^2 + dz_4^2\right) + X^3 \left(dz_5^2 + dz_6^2\right) , \\ \mathcal{A} &= A^1 \wedge dz_1 \wedge dz_2 + A^2 \wedge dz_3 \wedge dz_4 + A^3 \wedge dz_5 \wedge dz_6 . \end{aligned} \tag{44}$$

It is assumed that nothing depends on the coordinates z^i parametrizing the T^6, so we can regard ds^2_5, X^i and A^i as a five-dimensional metric, scalars, and vectors respectively. We assume that the scalars X^i obey a constraint

$$X^1 X^2 X^3 = 1. \tag{45}$$

This ensures that the T^6 has constant volume, which guarantees that the metric ds^2_5 is the five-dimensional Einstein-frame metric. The eleven-dimensional action reduces to the action of five-dimensional $U(1)^3$ supergravity:

$$I_5 = \frac{1}{16\pi G_5} \int \left(R \star 1 - G_{ij} dX^i \wedge \star dX^j - G_{ij} F^i \wedge \star F^j - \frac{1}{6} C_{ijk} F^i \wedge F^j \wedge A^k \right) , \tag{46}$$

where $G_{ij} = \frac{1}{2}\mathrm{diag}\left((X^1)^{-2}, (X^2)^{-2}, (X^3)^{-2}\right)$, $C_{ijk} = 1$ if (ijk) is a permutation of (123) and $C_{ijk} = 0$ otherwise, and the Maxwell field strengths are $F^i = dA^i$. In five dimensions, we can define conserved electric charges for asymptotically flat solutions by

$$\mathbf{Q}_i = \frac{1}{16\pi G_5} \int_{S^3} (X^i)^{-2} \star_5 F^i, \tag{47}$$

where the integral is evaluated at spatial infinity. Using standard techniques (*e.g.,* [19]), it can be shown that any appropriately regular solution of this theory satisfies the BPS inequality

$$M \geq |\mathbf{Q}_1| + |\mathbf{Q}_2| + |\mathbf{Q}_3| , \tag{48}$$

[9] More generally, one can compactify on a Calabi-Yau three-fold, leading to $N = 1$ supergravity in five-dimensions coupled to a certain number of vector multiplets (and hypermultiplets, which we need not consider). The results below are easily generalized to this case.

where M is the ADM mass. A solution is said to be supersymmetric if it saturates this inequality. Examining the eleven-dimensional field strength makes it clear that the electric charge $\mathbf{Q}_i$ that couples to F^i arises from M2-branes wrapped on the internal T^6, *e.g.,* F^1 is sourced by M2-branes wrapping the 12 cycle of T^6 etc. The charges are quantized in terms of the wrapping numbers of the M2-branes as

$$N_i = \left(\frac{4G_5}{\pi}\right)^{1/3} \mathbf{Q}_i \, . \tag{49}$$

We can also map these solutions to a U-duality frame that is convenient for microscopic analysis, namely, as solutions for a D1-D5-brane intersection with momentum running along their common direction. To this effect, dimensionally reduce the eleven-dimensional solution above on (say) the z^6 direction to give a solution of ten-dimensional type IIA supergravity. Performing T-dualities in the z^5, z^4, z^3 directions then gives a solution of type IIB supergravity with metric

$$ds^2 = -(X^3)^{1/2} ds_5^2 + (X^3)^{-3/2}\left(dz + A^3\right)^2 + X^1 (X^3)^{1/2} d\mathbf{z}_4^2, \tag{50}$$

where the T^5 is parametrized by the coordinates $z \equiv z^5$, $\mathbf{z}_4 \equiv (z^1, z^2, z^3, z^4)$. Note that the circle parametrized by z may be non-trivially fibered over the five-dimensional spacetime. We shall refer to this direction as the Kaluza-Klein circle. The other non-zero IIB fields are

$$e^{2\Phi} = \frac{X^1}{X^2}\,, \qquad F_{(3)} = \left(X^1\right)^{-2} \star_5 F^1 + F^2 \wedge \left(dz + A^3\right)\,, \tag{51}$$

where Φ is the dilaton, $F_{(3)}$ the Ramond-Ramond 3-form field strength, and $\star_5$ denotes the Hodge dual with respect to the five-dimensional metric. These formulae allow any solution of five-dimensional $U(1)^3$ supergravity to be uplifted to a solution of type IIB supergravity. Examining the RR 3-form reveals that the electric charges that couple to F^1 and F^2 arise from D5-branes wrapped on T^5 and D1-branes wrapped around the z-circle respectively. The appearance of A^3 in the metric reveals that F^3 is electrically sourced by momentum (P) around the KK z-circle.

4.2 Dipoles

The possible presence of dipoles on a black ring is most easily understood by recalling our discussion of the field of a circular string in Section 2. There we saw that a circular string gives rise to an electric field $B_{t\psi}$, whose magnetic dual A_ϕ is sourced by a circular distribution of magnetic monopoles. So, in addition to the charges $\mathbf{Q}_i$ defined in (47), the topology of a black ring allows to define 'dipole charges' q_i as we would do for a magnetic charge,

$$q_i = \frac{1}{2\pi}\int_{S^2} F^i\,, \tag{52}$$

by performing the integral on a surface S^2 that links the ring once, see figure 3. The field (6) corresponds to a unit dipole.

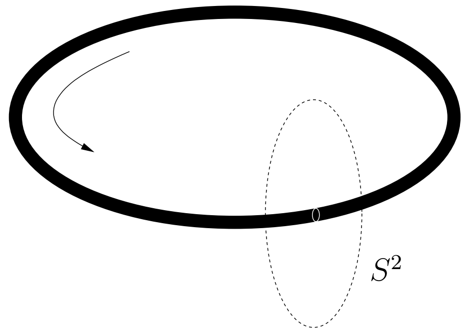

Figure 3: The dipole q_i measured from the magnetic flux of F^i across an S^2 that encloses a section of the string. An azimuthal angle has been suppressed in the picture, so the S^2 is represented as a circle.

However, even if there is a local distribution of charge, the total magnetic charge is zero: in order to compute the magnetic charge in five dimensions, one has to specify a two-sphere that encloses a point of the ring, *and* a vector tangent to the string. So there are opposite magnetic charges at diametrically opposite points of the ring, which justifies the analogy to a dipole. The magnetic charges can be completely annihilated by contracting the size of the ring to zero, so q_i is not a conserved quantity. Equivalently, in the dual electric picture, the field $B_{t\psi}$ is not associated to any conserved charge since ψ parametrizes a contractible cycle.

One might also attempt to define the dipole from the asymptotic fall off of the gauge fields, which, from (6), is

$$A_\phi \to q\frac{R^2 r_1^2}{(r_1^2+r_2^2)^2} + O(r_1^2+r_2^2)^{-2}\,. \tag{53}$$

However, if there are electric charges present, the magnetic dipole field will have a component induced by the rotation, which does not contribute to non-uniqueness. It is therefore preferable to use the definition (52) to characterize the dipole intrinsic to the source.

Dipole rings are therefore specified by the independent physical parameters (M, J, q_i). The q_i are non-conserved, classically continuous parameters. So they imply continuous violations of non-uniqueness in five dimensions. Rotating black rings with these dipoles were conjectured to exist in refs. [20, 21, 17], and were first found in [9].

In the embedding into M-theory described in the previous section, the black ring is made of M5-branes, with four worldvolume directions wrapping 4-cycles of the internal T^6 and the fifth direction being the circular ring direction. The charge q_i is then quantized as

$$n_i = \left(\frac{\pi}{4G_5}\right)^{1/3} q_i\,, \tag{54}$$

corresponding to the wrapping number of the M5-branes around the ring circle.

The solution for the dipole black rings of (46) contains, in addition to the parameters that the neutral solution already has, three new ones, μ_i. The three dipoles q_i are then functions of the μ_i, and when all $\mu_i = 0$ we recover the neutral solution (14). The explicit form of the metric is

$$\begin{aligned} ds_5^2 &= -\frac{F(y)}{F(x)}\frac{H(x)}{H(y)}\left(dt - C\,R\,\frac{1+y}{F(y)}\,d\psi\right)^2 \\ &\quad + \frac{R^2}{(x-y)^2}\,F(x)H(x)H(y)^2\left[-\frac{G(y)}{F(y)H(y)^3}d\psi^2 - \frac{dy^2}{G(y)} + \frac{dx^2}{G(x)} + \frac{G(x)}{F(x)H(x)^3}d\phi^2\right]. \end{aligned} \tag{55}$$

The functions F and G are as in (15), C is defined by (16), and

$$H(\xi) = \left[H_1(\xi)H_2(\xi)H_3(\xi)\right]^{1/3}\,, \tag{56}$$

with

$$H_i(\xi) = 1 - \mu_i\xi\,. \tag{57}$$

The gauge potentials are

$$A^i = C_i\,R\,\frac{1+x}{H_i(x)}\,d\phi\,, \tag{58}$$

and the scalars

$$X^i = \frac{H(x)H_i(y)}{H(y)H_i(x)}\,. \tag{59}$$

C_i is as in (16) but with $\lambda \to -\mu_i$. Since the field (58) is purely magnetic, it makes no contribution to the Chern-Simons term in the action.

The parameters λ and ν vary in the same ranges as in the neutral case (17), while

$$0 \le \mu_i < 1\,. \tag{60}$$

Expressions for the mass etc of this solution can be found in [9]. The addition of dipoles to a ring has several effects on its dynamics. The dipole increases the self-attraction between opposite points along the ring. For given black ring mass, if the dipole is non-vanishing the angular momentum is not only bounded below but also bounded above. Also, for fixed mass and angular momentum, the addition of a dipole reduces the area and the temperature of the black ring. When the three dipoles are present, the dipole ring has an outer and an inner horizon, and an upper bound on the magnitude of q is obtained when the two horizons coincide ($\nu = 0$) and the ring becomes extremal. This ring has a non-singular horizon of finite area and vanishing temperature. However, it is *not* supersymmetric, since in the absence of any conserved charges $\mathbf{Q}_i$ the BPS bound (48) cannot be saturated.

More surprisingly, dipoles feature in the first law

$$dM = \frac{T d\mathcal{A}_H}{4G_5} + \Omega dJ + \Phi^i d\mathbf{Q}_i + \phi^i dq_i\,. \tag{61}$$

Here ϕ^i is defined (up to convenient normalization) as the difference in the dipole potential at infinity and at the horizon. When $\mathbf{Q}_i = 0$ this equation was obtained in [9] from the explicit form of the dipole ring solutions. However, as noted in [22], the last term appears to be at odds with conventional derivations of the first law, which seem to allow only conserved charges into it. The resolution of this puzzle lies in the impossibility to define the dipole potential, using a single patch, such that it is simultaneously regular at the rotation axis $y = -1$ and at the horizon. As a result, a new surface term enters the first law, giving precisely (61) [22] (see also [23]).

5 Supersymmetric Black Rings

5.1 The solution and its properties

The possible existence of supersymmetric black rings was suggested in [24, 25] based on thought experiments involving supersymmetric black holes and supertubes. The subsequent discovery of supersymmetric black ring solutions grew out of parallel studies of charged black rings [17, 26] and of a program to classify supersymmetric solutions of five-dimensional $N = 1$ supergravity. It turns out that there is a canonical form for such solutions [27, 28], with the necessary and sufficient conditions for supersymmetry reducing to simple-looking equations on a four-dimensional "base space". The first supersymmetric black ring solution [29] was obtained by solving these equations for minimal 5D supergravity, taking the base space to be flat space written in ring coordinates as in (10). This was subsequently generalized to $U(1)^3$ supergravity by three independent groups [30, 31, 32].[10] The solution is:

$$\begin{aligned} ds_5^2 &= -(H_1H_2H_3)^{-2/3}(dt+\omega)^2 + (H_1H_2H_3)^{1/3} d\mathbf{x}_4^2\,, \\ A^i &= H_i^{-1}(dt+\omega) + \frac{q_i}{2}\left[(1+y)d\psi + (1+x)d\phi\right], \\ X^i &= H_i^{-1}(H_1H_2H_3)^{1/3}\,, \end{aligned} \tag{62}$$

where $d\mathbf{x}_4^2$ is the flat base space that we encountered in (10), the functions H_i are

$$\begin{aligned} H_1 &= 1 + \frac{Q_1 - q_2q_3}{2R^2}(x-y) - \frac{q_2q_3}{4R^2}(x^2-y^2), \\ H_2 &= 1 + \frac{Q_2 - q_3q_1}{2R^2}(x-y) - \frac{q_3q_1}{4R^2}(x^2-y^2), \end{aligned} \tag{63}$$

[10] In fact, the same method yields black ring solutions of $U(1)^n$ supergravity [31, 32].

$$H_3 = 1 + \frac{Q_3 - q_1 q_2}{2R^2}(x-y) - \frac{q_1 q_2}{4R^2}(x^2-y^2),$$

and $\omega = \omega_\phi d\phi + \omega_\psi d\psi$ with

$$\begin{aligned} \omega_\phi &= \frac{1}{8R^2}(1-x^2)\left[q_1 Q_1 + q_2 Q_2 + q_3 Q_3 - q_1 q_2 q_3 \,(3+x+y)\right], \\ \omega_\psi &= -\frac{1}{2}(q_1+q_2+q_3)(1+y) + \frac{1}{8R^2}(y^2-1)\left[q_1 Q_1 + q_2 Q_2 + q_3 Q_3 - q_1 q_2 q_3 \,(3+x+y)\right]. \end{aligned} \tag{64}$$

The coordinate ranges are as they would be for the metric (10).

The solution depends on the seven parameters q_i, Q_i and R. The q_i are dipole charges defined by (52) with the integral taken over a surface of constant t, y and ψ. Q_i are proportional to the conserved charges:

$$\mathbf{Q}_i = \frac{\pi}{4G_5} Q_i, \tag{65}$$

and R is a length scale corresponding to the radius of the ring with respect to the base space metric. In the limit $R \to \infty$ with the charge densities Q_i/R fixed, the solution reduces to a black string solution obtained in [25] (the change of coordinates and limit required are essentially the same as in the neutral case (19)). Supersymmetry implies that the mass is fixed by saturation of the BPS inequality (48). The three Killing fields generate a $\mathbf{R} \times U(1) \times U(1)$ isometry group, just as for nonsupersymmetric black rings.

The most obviously novel feature of this solution is the fact that $\omega_\phi \neq 0$: the solution rotates in both the ϕ and ψ directions. The angular momenta are

$$\begin{aligned} J_\phi &= \frac{\pi}{8G_5}\left(q_1 Q_1 + q_2 Q_2 + q_3 Q_3 - q_1 q_2 q_3\right), \\ J_\psi &= \frac{\pi}{8G_5}\left[2R^2(q_1+q_2+q_3) + q_1 Q_1 + q_2 Q_2 + q_3 Q_3 - q_1 q_2 q_3\right]. \end{aligned} \tag{66}$$

Note that the parameter R is determined by $J_\psi - J_\phi$ and the dipoles.

Many supersymmetric solutions suffer from causal pathologies such as closed causal curves (CCCs) [27]. The necessary and sufficient condition for the above solution to be free of closed causal curves for $y \geq -\infty$ is [31]

$$2q^2 L^2 \equiv 2\sum_{i<j} \mathcal{Q}_i q_i \mathcal{Q}_j q_j - \sum_i \mathcal{Q}_i^2 q_i^2 - 4R^2 q^3 \sum_i q_i \geq 0, \tag{67}$$

where we have defined

$$q = (q_1 q_2 q_3)^{1/3}, \qquad \mathcal{Q}_1 = Q_1 - q_2 q_3, \qquad \mathcal{Q}_2 = Q_2 - q_3 q_1, \qquad \mathcal{Q}_3 = Q_3 - q_1 q_2. \tag{68}$$

If the inequality in (67) is strict then the solution has an event horizon at $y \to -\infty$ [31, 32]. Just as for non-supersymmetric rings, ψ and t are not good coordinates on the horizon and have to be replaced by new coordinates ψ' and v. For supersymmetric rings, the ϕ rotation implies that ϕ is also not a good coordinate, however $\chi \equiv \phi - \psi$ is. The geometry of a spacelike section of the horizon is

$$ds_H^2 = L^2 d\psi'^2 + \frac{q^2}{4}\left(d\bar{\theta}^2 + \sin^2\bar{\theta} d\chi^2\right), \tag{69}$$

where $x = \cos\bar{\theta}$ as before. So the horizon is geometrically a product of a circle of radius L and a two-sphere of radius $q/2$. The entropy can be calculated from the horizon area:

$$S = \frac{\pi^2 L q^2}{2G_5} = 2\pi\sqrt{\frac{c\hat{q}_0}{6}}, \tag{70}$$

where, using the quantized charges N_i and n_i in (49), (54),

$$c = 6n_1n_2n_3, \tag{71}$$

and

$$\hat{q}_0 = \frac{n_1n_2n_3}{4} + \frac{1}{2}\left(\frac{N_1N_2}{n_3} + \frac{N_2N_3}{n_1} + \frac{N_1N_3}{n_2}\right) - \frac{1}{4n_1n_2n_3}\left[(N_1n_1)^2 + (N_2n_2)^2 + (N_3n_3)^2\right] - J_\psi. \tag{72}$$

The reason for writing the entropy is this rather odd form will become apparent when we discuss the microscopic interpretation of supersymmetric black rings.

Finally, we note that the angular velocities of the horizon of a supersymmetric black ring vanish, as is necessarily the case for a supersymmetric, asymptotically flat black hole [33].

5.2 Non-uniqueness

Before the discovery of supersymmetric black rings, the only know supersymmetric black hole solution of five-dimensional supergravity was the so-called BMPV black hole [34]. This solution can be obtained as a limit of the supersymmetric black ring solution. To this end, consider the change of coordinates

$$\rho\cos\Theta = r_1, \qquad \rho\sin\Theta = r_2\,, \tag{73}$$

where r_1, r_2 were defined in (8) and $0 \le \rho < \infty$, $0 \le \Theta \le \pi/2$. In these coordinates the flat base space metric is

$$d\mathbf{x}_4^2 = d\rho^2 + \rho^2(d\Theta^2 + \sin^2\Theta d\psi^2 + \cos^2\Theta d\phi^2)\,. \tag{74}$$

The form of the above solution in these coordinates is given in [31]. To obtain the BMPV solution we take the limit $R \to 0$ with the coordinates and other parameters held fixed. The solution becomes

$$H_i = 1 + \frac{Q_i}{\rho^2}, \qquad A^i = H_i^{-1}(dt + \omega), \tag{75}$$

$$\omega_\phi = \frac{4G_5 J}{\pi}\frac{\cos^2\Theta}{\rho^2}\,, \qquad \omega_\psi = \frac{4G_5 J}{\pi}\frac{\sin^2\Theta}{\rho^2}\,, \tag{76}$$

where

$$J = \frac{\pi}{8G_5}\left[q_1 Q_1 + q_2 Q_2 + q_3 Q_3 - q_1 q_2 q_3\right]\,. \tag{77}$$

This solution is determined by four parameters: Q_i and J. The former retain their interpretation as conserved M2-brane charges. The latter determines the angular momenta: $J_\phi = J_\psi = J$. The solution has an event horizon at $\rho = 0$ of topology S^3. The solution is much more symmetrical than the supersymmetric black ring solution: it has isometry group $\mathbf{R} \times U(1) \times SU(2)$ [33], although this is not manifest in the above coordinates.

Since the angular momenta of the BMPV black hole are equal and those of a supersymmetric black ring are always unequal, it follows that one can always distinguish these two types of supersymmetric black hole by comparing their conserved charges. Nevertheless, the conserved charges of a supersymmetric black ring can still be made arbitrarily close to those of a BMPV black hole by taking R small enough.

Although supersymmetric black rings cannot carry the same conserved charges as a BMPV black hole, the fact that dipole charges are required to specify them entails a violation of black hole uniqueness in exactly the same way as for non-supersymmetric dipole rings. Seven parameters are required to specify the solution but there are only five independent conserved charges, namely $\mathbf{Q}_i$, J_ϕ and J_ψ. Hence there is a continuous violation of black hole uniqueness even for supersymmetric black holes. However, if one takes charge quantization into account then this violation of uniqueness is rendered finite [31].

A more extreme violation of black hole uniqueness was proposed in [30]. It was realized in [30] that the problem of finding supersymmetric solutions can be reduced to specifying appropriate sources for certain harmonic functions on the base space. Physically, these sources describe M2-branes with both worldvolume directions wrapped on the internal torus, and M5-branes with four worldvolume directions wrapped on the torus, *i.e.,* “M2-particles” and “M5-strings” in the five noncompact directions. Choosing the sources to correspond to a circular loop of M5-strings with a constant density of M2-particles distributed around the string leads directly to the supersymmetric black ring solution above [30]. One can also construct more general solutions for which the loop of M5 branes is not (geometrically) a circle or the density of M2 branes not constant [30, 35]. However, it turns out that such non-uniform solutions do not admit smooth horizons so they do not describe black holes [36]. Nevertheless, one can still assign an entropy to them formally and it may be possible to make sense of them in string theory.

5.3 Multi-ring solutions

In $D = 4$, there exist solutions of Einstein-Maxwell theory describing a static superposition of several extremal Reissner-Nordstrom black holes [37]. Physically, these solutions exist because there is a cancellation of gravitational attraction and electrostatic repulsion between the holes. This is often the case for supersymmetric systems (of which these solutions are an example [19]). It is therefore natural to ask whether there exist solutions describing multiple supersymmetric black rings or superpositions of supersymmetric black rings with BMPV black holes. It turns out that such solutions do indeed exist [38, 32].

The simplest way to understand these solutions is to imagine constructing them using the method of [30] and prescribing sources corresponding to multiple rings. However, in order to obtain the corresponding solutions in a form sufficiently explicit to analyze in detail, a different approach is required. Following [38] we can introduce new coordinates $\chi = \phi - \psi$, $\alpha = -\phi - \psi$, $\bar{\theta} = \Theta/2$ and $\bar{r} = \rho^2/(4\ell)$, where Θ and ρ are defined in (73) and ℓ is an arbitrary length. The flat metric (74) becomes

$$d\mathbf{x}_4^2 = H^{-1}\left(d\alpha + \cos\bar{\theta}d\chi\right)^2 + H\left(d\bar{r}^2 + \bar{r}^2 d\bar{\theta}^2 + \bar{r}^2\sin^2\bar{\theta}d\chi^2\right), \tag{78}$$

where $H = \ell/\bar{r}$. This flat metric is a special case of a more general family of Ricci-flat metrics known as *Gibbons-Hawking* metrics [39]:

$$ds^2 = H^{-1}\left(d\alpha + \mathbf{w}\right)^2 + H\delta_{ij}dx^i dx^j, \tag{79}$$

with $\partial/\partial\alpha$ a Killing field, $\mathbf{w} \equiv w_i dx^i$ obeys $\nabla \times \mathbf{w} = \nabla H$ (where $\nabla_i = \partial_i$), which implies that H is harmonic on $\mathbf{R}^3$: $\nabla^2 H = 0$. It was shown in [27, 32] that the general supersymmetric solution with a Gibbons-Hawking base space for which $\partial/\partial\alpha$ extends to a space-time symmetry is specified by harmonic functions on $\mathbf{R}^3$ (of which H is one). The supersymmetric black ring solution discussed above is such a solution. It turns out that, with the exception of H, the harmonic functions for black ring solutions all have a single pole at a certain point on the negative z-axis in $\mathbf{R}^3$ with the position related to the "radius" R of the ring.

The multi-ring solutions of [38, 32] are obtained by taking the harmonic functions to have more general sources[11] with poles at several points in $\mathbf{R}^3$, each corresponding to a different ring. If these poles all lie along the z-axis then the resulting solution preserves the same $U(1) \times U(1)$ symmetry on the base space as a single black ring. This implies that, on the base space, each ring corresponds to a circle centred on the origin and lying either in the 12 plane or the 34 plane in the coordinates of (1). (Whether it is the 12 plane or the 34 plane is dictated by whether the relevant pole in the harmonic functions is on the negative

[11] Although still with $H = \ell/\bar{r}$, *i.e.,* a flat base space.

or positive z-axis). The solutions admit regular event horizons and appear free of causal pathologies provided certain restrictions on the parameters are satisfied.[12]

More general solutions, in which the poles do not all lie on the z-axis can also be constructed [38, 32]. These preserve only a single $U(1)$ symmetry on the base space and correspond to rings centered on the origin but no longer restricted to the 12 or 34 planes. They appear to be free of pathologies close to individual rings but it is not known what extra conditions are required for these solutions to be well-behaved globally.

Note that if one takes a multi-ring system and shrinks one of the rings down to zero radius then one obtains a solution describing a BMPV black hole sitting at the common centre of the remaining rings [38, 32]. For the case of a single ring and a single BMPV black hole, such a solution can be written down using the original ring coordinates of (10) [30].

Finally, we note that the above construction can be generalized by replacing the flat base space with a more general Gibbons-Hawking space. A particularly interesting choice is (self-dual, Euclidean) Taub-NUT space, which corresponds to $H = 1 + \ell/\bar{r}$. This has the same topology as $\mathbf{R}^4$ but differs geometrically. Surfaces of constant $\bar{r}$ have S^3 topology but, viewing S^3 as a S^1 bundle over S^2, the radius of the S^1 approaches a constant as $\bar{r} \to \infty$ whereas the radius of the S^2 grows as $\bar{r}$. Hence solutions with Taub-NUT base space obey Kaluza-Klein, rather than asymptotically flat, boundary conditions. The method of [27, 32] can be used to obtained solutions describing BMPV black holes [40] and (multiple concentric) supersymmetric black rings [41, 42, 43] in Taub-NUT space. After Kaluza-Klein reduction, the latter solutions correspond to the 4D multi-black hole bound states obtained in [44]. This "4D-5D connection" allows one to extend recently discovered relations between 4D black holes and topological string theory to 5D black holes [40].

6 Microscopics of Black Rings

The microscopic description of black holes in string theory is typically based on the dynamics of a configuration of branes that has the same set of charges as the black hole. Black rings can carry both conserved and dipole charges: depending on which of the two sets one puts the stress on, the description is rather different. But they are related: the conserved-charge-based description is the ultraviolet (UV) completion of the dipole-based one, which describes only the physics of the system at the lowest energies, *i.e.,* the infrared (IR).

Both theories are two-dimensional sigma-models, and in the extremal limit where the

[12] A possible explanation for the these restrictions is the following. Supersymmetry guarantees cancellation of gravitational and electromagnetic forces, but not of forces arising from spin-spin interactions. The extra condition might arise from requiring that these interactions vanish.

momentum is chiral the entropy follows from the Cardy formula

$$S = 2\pi\sqrt{\frac{c\hat{q}_0}{6}}\,. \tag{80}$$

The central charge, c, and the momentum available to distribute among chiral oscillators, $\hat{q}_0$, differ in each description (even if we refer to the same object). The two-dimensional sigma-model can be regarded as an "effective string" and the descriptions differ in what we take this string to be:

- The IR theory has the effective string extending along the S^1 direction of the ring, so we view the ring as a circular string. This was first proposed and applied to extremal non-supersymmetric black rings in [9]. Its application to supersymmetric black rings gives an impressive match of the statistical and Bekenstein-Hawking entropies [45, 46].
- In the UV theory the effective string direction extends along a sixth-dimension orthogonal to the ring. Thus we view the ring as a tube—more properly, a supertube, or an excitation of it. This was first proposed for two-charge black rings in [17], and developed for supersymmetric three-charge black rings in [45].

As we will see, each description has its virtues and shortcomings.

6.1 IR theory: Black rings as circular strings

This is based on the worldvolume theory of the branes that carry the dipoles, which have one worldvolume direction along the ring circle. The microscopic theory for the straight string limit of the ring is then applied to a circular ring of finite radius. So far, none of the proposed IR theories can distinguish between a ring and a KK compactified string, *i.e.,* they work to the extent that finite radius corrections to the entropy cancel out.

For a black ring with a finite horizon in the extremal limit, a convenient description is obtained in terms of a triple intersection of M5-branes, with momentum running along the ring, as we have seen in sections 4 and 5. If the six compact directions of space are small, then the low energy dynamics is described by a $(0,4)$-supersymmetric $1+1$ sigma-model at the intersection of the branes. Following [47], the central charge c of the theory is given by the number of moduli that parametrize the deformations of the (smoothed) intersection of branes, and is proportional to the number of branes of each kind. A detailed calculation gives

$$c_{IR} = 6n_1n_2n_3\,. \tag{81}$$

The supergravity and sigma-model descriptions are valid when the volume of the six-dimensional internal space (in 11D Planck units) is, respectively, $V_6 \ll c_{IR}$, or $V_6 \gg c_{IR}$. Additionally, one requires the radius of the ring $R \gg V_6^{1/6}$ (to reduce to a sigma-model), and $V_6 \gg 1$ (to neglect quantum corrections to supergravity) [47].

Conserved charges corresponding to M2 branes are obtained by turning on worldvolume fluxes. For the moment, we set these fluxes to zero. Then the ring carries only M5 dipoles and angular momentum J_ψ, which in the absence of fluxes is identified with the effective string momentum $\hat{q}_0$. At finite radius, this is not a supersymmetric solution even if the momentum is chiral. This corresponds to a black ring solution that is the extremal limit of the dipole ring of (55). It is straightforward to check that the resulting microscopic degeneracy formula

$$S = 2\pi\sqrt{n_1 n_2 n_3 J_\psi} \tag{82}$$

correctly describes the entropy of the black ring in the straight string limit $R \to \infty$, which is no more than the known entropy match for the black string [47]. Ref. [9] showed that the model does even better, since it also captures correctly the leading corrections of the black ring entropy in a $1/R$ expansion.[13] These corrections already include the effects of the self-interaction among different points along the ring —an effect of the finite ring radius. These effects become too strong in subleading corrections, however, and it is not clear how to account for them.

Supersymmetric black rings provide a better behaved system: finite radius effects appear to be absent from the entropy at any radius, although this has not been explained microscopically and so the conclusions are less rigorous than would be desired. To saturate the BPS bound, supersymmetric black rings necessarily carry conserved M2 charges, with integer brane numbers N_i; in the microscopic picture, we turn on fluxes on the worldvolume of the M5 branes. These fluxes also give rise to momentum zero-modes that contribute to the total momentum q_0, so this is no longer equal to the momentum available to non-zero-mode oscillators $\hat{q}_0$. Instead the relationship is [47, 46]

$$\hat{q}_0 = q_0 + \frac{1}{2}\left(\frac{N_1 N_2}{n_3} + \frac{N_2 N_3}{n_1} + \frac{N_1 N_3}{n_2}\right) - \frac{1}{4 n_1 n_2 n_3}\Big((N_1 n_1)^2 + (N_2 n_2)^2 + (N_3 n_3)^2\Big) + \frac{n_1 n_2 n_3}{4}\,. \tag{83}$$

Comparing to (72) we see that the choice $q_0 = -J_\psi$ yields a perfect match of the microscopic entropy (80) to the Bekenstein-Hawking entropy (70). The last term in (83) is a zero-point correction to the momentum and as we see is necessary to find perfect agreement with the entropy of the black ring.

In the analysis of [45] the identification of parameters is slightly different. It can be

[13]Bear in mind that R is not an independent parameter but is fixed by the other charges so in a $1/R$ expansion we take some combination of charges to be large.

checked that $\hat{q}_0$ in (72) can alternatively be written as

$$\hat{q}_0 = -J_\psi + J_\phi + \frac{1}{2}\left(\frac{\mathcal{N}_1\mathcal{N}_2}{n_3} + \frac{\mathcal{N}_2\mathcal{N}_3}{n_1} + \frac{\mathcal{N}_1\mathcal{N}_3}{n_2}\right) - \frac{1}{4n_1n_2n_3}\Big((\mathcal{N}_1 n_1)^2 + (\mathcal{N}_2 n_2)^2 + (\mathcal{N}_3 n_3)^2\Big)\,, \tag{84}$$

where $\mathcal{N}_1 = N_1 - n_2n_3$, and the obvious permutations for $\mathcal{N}_{2,3}$. Ref. [45] proposes that $\mathcal{Q}_i$, instead of Q_i, is the quantity to be identified with the M2-brane charge at the source, so the actual microscopic M2-brane numbers are $\mathcal{N}_i$ instead of N_i. One must note, though, that there is no known invariant definition of charge that justifies this choice [36]. If one then equates $q_0 = -J_\psi + J_\phi$, and ignores the zero-point term in (83), the entropy is reproduced. This match uses crucially the fact that the entropy is given by the quartic invariant of the U-duality group E_7 of M-theory on $\mathrm{CY}_3 \times S^1$ at low energies, *i.e.,* the theory that describes a KK compactified black string, so finite radius effects are again ignored. A possible rationale why the entropy of the compactified KK string and the black ring should agree is given in [43] via consideration of black rings in Taub-NUT. By varying the modulus corresponding to the KK radius, one can interpolate between the compactified black string and the black ring. Since the entropy is moduli-invariant it should remain the same for both limits.

It is striking, and not at all well understood, that these two different calculations reproduce exactly the entropy of the black ring. Although the match between the entropies is remarkable, both of these microscopic descriptions clearly leave many points obscure by being unable to say anything about finite radius effects. Besides the problems already mentioned, the calculation in [46] does not explain why the M2 charges Q_i are bounded below by the dipoles so that $\mathcal{Q}_i > 0$. Moreover, the microscopic picture would seem to place no restrictions on the angular momentum J_ψ nor the ring radius R. However, these cannot be varied independently if the ring is to remain in equilibrium. The role of the second angular momentum (J_ϕ in [46], $J_\psi + J_\phi$ in [45]), which does not appear anywhere in the entropy formulas, is also unclear (some ideas are discussed in [31, 48]). Note that this second angular momentum is fixed by the other charges, and it should be possible to calculate it in the microscopic theory. The result should agree with supergravity because angular momentum is quantized and hence not renormalized. However, the proposals of [45, 46] would appear to ascribe a vanishing value to this angular momentum.

But perhaps a more important deficiency, inherent to the description, is that it does not allow to say anything about the microscopic significance of black hole non-uniqueness. In a sense, the dipole-based IR theory looks too closely at the ring, and by focusing on the string-like aspects of the ring, it cannot describe the spherical black hole. To be able to view both black objects from a unified perspective, we have to step back and observe them from a greater distance (so, by AdS/CFT duality, we go to the ultraviolet), where the conserved charges play the dominant role.

6.2 UV theory: Black rings as supertubes

The dynamics is now determined by the worldvolume theory on the branes that carry the conserved charges. In principle this CFT can describe both spherical black holes and black rings with the same charges as different phases of the theory, with the dipoles acting as order parameters. Depending on the phase, the theory has a different flow to the IR. While the spherical black hole phase is exactly conformal, the black ring induces a non-trivial flow to the theory of the previous subsection. One can check that the central charge of the IR theory is indeed less than that of the UV theory [45, 49], which is quantitative evidence in favour of this picture. The supersymmetries in one of the chiral sectors of the (4,4) UV theory are broken along the flow, so the IR theory has (0,4) supersymmetry.

It is convenient to pass to a different U-duality frame. The five-dimensional supersymmetric black hole is best understood by first uplifting it to a black string in six dimensions, and viewing it as an intersection of D1 and D5-branes that carry momentum along this sixth, common direction, as discussed near the end of subsec. 4.1. Again, at low energies the dynamics is captured by a $1 + 1$ CFT with central charge

$$c_{UV} = 6N_1N_2 \tag{85}$$

where N_1, N_2 are the numbers of D1 and D5 branes.

This sigma-model CFT is well-understood only at a point in moduli space where its target space is a symmetric orbifold of N_1N_2 copies of the internal four-manifold. Roughly speaking, at this point the theory is 'free'. The supergravity description, instead, corresponds to a deformation of the theory away from it (a 'strong coupling' regime). However, the symmetric orbifold theory seems to capture correctly many of the features of black holes with D1-D5 charges.

At the orbifold point the CFT contains twisted sectors. Pictorially, the maximally twisted sector corresponds to a long effective string, of length N_1N_2 times the length of the physical circle that the string wraps (which we take to be equal to one). The energy gap of momentum excitations is then smallest $\sim 1/N_1N_2$. The untwisted sector can be regarded as containing a number N_1N_2 of short effective strings, each of unit length. Momentum excitations have large gap $\sim O(1)$. There are also partially twisted sectors.

To describe the spherical black hole with D1-D5-P charges, we put the string in the maximally twisted sector. The linear momentum P, which in this case is one of the three conserved charges, is carried by both bosonic and fermionic excitations along the effective string. The fermionic excitations can also carry a polarization in the transverse directions (as R-charge in the CFT). If there are q_J such oscillators polarized in the same direction, they give rise to a self-dual angular momentum

$$J_\psi = J_\phi = J = \sqrt{\frac{c_{UV} q_J}{6}}\,. \tag{86}$$

The projection of oscillators onto a given polarization restricts the phase space so the $\hat{q}_0$ that enters the Cardy formula is smaller than the units of momentum $q_0 = N_3$ that correspond to P,

$$\hat{q}_0 = q_0 - q_J = q_0 - \frac{6J^2}{c_{UV}}\,. \tag{87}$$

This reproduces the entropy $S = 2\pi\sqrt{N_1 N_2 N_3 - J^2}$ of the supersymmetric rotating BMPV black hole and provides a detailed account of its properties [34].

To understand how black rings fit into this theory, observe that there is another way in which angular momentum can be carried by the D1-D5 system. Each individual effective string has a fermionic ground state that can be polarized to carry angular momentum 1 (*i.e.,* $(1/2, 1/2)$ of the rotation group $SU(2) \times SU(2) \sim SO(4)$). In the untwisted ground state there are $N_1 N_2$ such short strings, which can therefore carry angular momentum

$$J_\psi = N_1 N_2\,, \qquad J_\phi = 0\,. \tag{88}$$

In this case angular momentum is present even in the absence of momentum excitations. This ground state, which is a unique microstate, corresponds to a class of systems generically known as *supertubes* [50], since their spacetime realization is typically in terms of tubular configurations of branes, in the present case a tube made of a single Kaluza-Klein monopole, $n_3 = 1$. If there are several such monopoles, $n_3 > 1$, then we are in a sector with $N_1 N_2/n_3 = c_{UV}/6n_3$ strings of length n_3 and the angular momentum of the supertube is

$$J_\psi = \frac{c_{UV}}{6n_3}. \tag{89}$$

Supertubes have the right topology to be identified as constituents of black rings and therefore provide string theory with the structure required to accommodate different black objects with the same conserved charges [17]. On the other hand, in order to obtain a macroscopic degeneracy it seems necessary to have a 'long-string' which contains a thermal ensemble of thinly-spaced (small gap) momentum excitations in much the same way as the BMPV black hole. Thus it is natural to propose that the CFT of three-charge black rings decomposes into two sectors, twisted and untwisted, with central charges c_1, c_2 adding to $c_{UV} = c_1 + c_2 = 6N_1 N_2$ [45]. The twisted "BMPV string" carries the momentum charge and hence provides the entropy and the J_ϕ component of the angular momentum. The untwisted "supertube string" accounts for the anti-self-dual component of the angular momentum,

$J_\psi - J_\phi$, and is responsible for the tubular structure of the configuration. Then the natural choice for the central charge of the supertube string is, from (89), $c_2 = 6n_3(J_\psi - J_\phi)$.

This is an appealing, simple picture. If we take the BMPV string to be maximally rotating, $q_J = N_3$, then its angular momentum is

$$J_\phi = \sqrt{\frac{c_1}{6} N_3} = \sqrt{\left[N_1 N_2 - n_3(J_\psi - J_\phi)\right] N_3}\,, \tag{90}$$

and the entropy vanishes (to leading order). Some zero-area three-charge black rings (those with $\mathcal{N}_1\mathcal{N}_2 = n_3(J_\psi - J_\phi)$, so $c_2 = 6\mathcal{N}_1\mathcal{N}_2$) appear to be accurately described by these formulae. However, the picture becomes problematic for configurations with non-zero entropy. One apparent difficulty is that in this description the transition from the black ring to the BMPV black hole is smooth —one simply eliminates the supertube sector, which does not contribute any entropy— in contradiction with the finite jump in the area of the corresponding supergravity solutions. Indeed, when the rotation of the BMPV string is less than maximal and there is a degeneracy of states, the choice of c_2 for the supertube string leaves too much central charge for the BMPV string and the microscopic entropy is too large. This can be adjusted *ad hoc* but it is unclear how to justify the larger value of c_2 that is needed to reproduce the entropy of black rings [45].

7 Other related developments

7.1 Non-supersymmetric black rings, and the most general black ring solution

The most general solution constructed so far for non-supersymmetric black rings is a seven-parameter family of non-supersymmetric black rings which have three conserved charges, three dipole charges, two unequal angular momenta, and finite energy above the BPS bound [26]. They have been found by solution-generating techniques (boosts and U-dualities) applied to the five-dimensional dipole black ring of [9] (see also [18, 17]). They are needed in order to understand the thermal excitations of two- and three-charge supertubes.

The supersymmetric limit of these solutions can only reproduce a supersymmetric ring with three charges and at most two dipoles. A larger family of non-supersymmetric black rings with nine-parameters $(M, J_\psi, J_\phi, Q_{1,2,3}, q_{1,2,3})$ is expected to exist, such that the general solutions of [29, 30, 31, 32] are recovered in the supersymmetric limit. The nine parameters would yield three-fold continuous non-uniqueness, the same as in the dipole rings of [9], furnished by the non-conserved dipole charges $q_{1,2,3}$.

In the limit in which the charges and dipoles vanish, this 9-parameter solution would reduce to a 3-parameter vacuum solution generalizing that of [1]. The third parameter would be angular momentum J_ϕ on the S^2: this would be a "doubly spinning" vacuum black ring. The most promising method of obtaining this vacuum solution directly appears to be the solution-generating techniques of [51]. However, these methods seem to involve considerably more guesswork than the relatively straightforward construction of the Kerr solution using analogous methods in four dimensions.

It has been argued that the most general black ring solution should be considerably larger than the 9-parameter family just discussed [52]. Toroidal compactification of eleven dimensional supergravity to five dimensions yields $N = 4$ supergravity, with 27 vectors and hence 27 independent electric charges. U-duality can be used to eliminate 24 of these charges, so there is no loss of generality in considering black holes carrying just 3 charges [53], corresponding to our Q_i. However, a general black ring should also carry 27 dipoles. Using U-duality, the best that can be done in general is to map this to a generating solution with 3 charges and 15 dipoles. This indicates that the most general supersymmetric black ring solution should have 19 parameters, and the most general non-supersymmetric black ring will have 21 parameters. Constructing even the supersymmetric solution will require some new ideas since the general form of supersymmetric solutions of $N = 4$ supergravity is a lot more complicated than that of the $N = 1$ $U(1)^3$ theory discussed above.

7.2 'Small' black rings

Supersymmetric rings with only two charges, and hence one dipole, have a naked singularity instead of a horizon. However, the microscopic theory still assigns them a finite entropy[14]

$$S = 4\pi\sqrt{N_1 N_2 - n_3 J}\,. \tag{91}$$

This is the degeneracy of fluctuations around the circular shape of a supertube with less than maximal angular momentum, $J < N_1 N_2/n_3$. In the terms used in the previous section, it can be described as a Bose-Einstein condensate of J short strings of length n_3 (the supertube), which account for the angular momentum, plus a thermal ensemble of strings with degeneracy (91) [54, 55].

Following the calculations performed for 'small black holes' [56], it has been possible to work out the regularization of the singularity by higher-derivative corrections to the low energy effective action in string theory. This requires the use of techniques developed for four-dimensional $N = 4$ supergravity, so the ring is compactified to four dimensions by

[14]When the internal space is $K3 \times S^1$. For T^5 the numerical prefactor is different and there is a mismatch with the gravitational entropy.

putting it on a Taub-NUT space. The Bekenstein-Hawking-Wald entropy of the corrected horizon can then be shown to reproduce (91) [55].

A simple model for the dynamical appearance of the Bose-Einstein condensate has been given in [57], where it is proposed that the creation operators for short strings of length n_3 be assigned a dipole $1/n_3$. The one-point functions of operators in the CFT dual to the small black ring are non-trivial and match well with the above description.

7.3 Non-singular microstates and foaming black rings

A main line of research that has provided a parallel motivation for much of the work on black rings and related solutions is the "fuzzball" proposal for the fundamental structure of black holes [58]. According to this proposal, there should be some U-duality frame in which black hole microstates admit a geometric description in terms of non-singular, horizon-free supergravity solutions. The black hole is to be regarded as an effective geometry describing an ensemble of microstates.

This programme has inspired the search for gravitating "microstate" solutions which are horizon-free and non-singular and which have the same charges as a given black hole. In five dimensions, the complete class of such (supersymmetric) solutions for the two-charge case is known [59] and one can indeed associate a solution with each microstate of the underlying CFT. In this 2-charge case, the system does not possess enough entropy to give rise to a macroscopic horizon so 2-charge black holes do not exist (although higher-derivative corrections can give rise to "small black holes"). Nevertheless, the microstate solutions do appear surprisingly "black-hole-like", in particular they exhibit a "throat" region which closes off at a radius set by the charges. If one computes the Bekenstein-Hawking entropy associated with this radius then it agrees (up to a factor) with the entropy obtained from CFT.

Some 3-charge microstate solutions are known [60] although too few to make a non-vanishing contribution to the entropy. Progress towards the construction of 3-charge solutions describing black ring microstates has been made by adding a small amount of momentum P as a perturbation to 2-charge supertube solutions [61].

A large class of 3-charge solutions proposed as black hole and ring microstate solutions has been constructed and studied in refs. [62, 63]. Refs. [63] apply the techniques described in subsec. 5.3 to multi-center Gibbons-Hawking base spaces with poles of positive and negative residues. The solutions exhibit a rich topological structure related to the "resolution" of dipole sources into fluxes along new internal cycles. However, in contrast to the solutions of the previous paragraph, in these cases a mapping between these supergravity solutions and dual CFT states has not been identified.

8 Outlook

8.1 Stability

Thus far we have left entirely aside the important issue of the classical dynamical stability of black rings. Supersymmetry should ensure that supersymmetric black rings are stable to quadratic fluctuations. Presumably, near-supersymmetric black rings are also stable within a range of parameters.

Vacuum black rings present a much more difficult case, since the study of their linearized perturbations appears to be much harder than in the case of Myers-Perry black holes:[15] even the equation for massless scalar fields in the presence of the black ring does not appear to be separable. Therefore, the study of gravitational perturbations around such backgrounds might require evolving the equations of motion for such perturbations numerically. This does not sound conceptually challenging so it will be interesting to see whether progress can be made this way.

For the time being, some qualitative and semi-quantitative arguments have been advanced. The black ring at the cusp between the thin and fat black ring branches must certainly be unstable: by throwing at it any small amount of matter that adds mass but not angular momentum, there is no other black ring that the system can evolve into and therefore it must backreact violently [65]. Qualitatively, it is expected that thin black rings suffer from the Gregory-Laflamme instability [66], which would grow lumps on the ring and whose evolution is at present uncertain [1, 67, 68].

A study of the topology of the phase diagram of black rings and MP black holes indicates that, precisely at the cusp, at least one unstable mode is added when going from thin to fat black rings, *i.e.,* the latter should be more unstable [69]. This is consistent with an analysis of radial perturbations [68], against which all fat black rings appear to be unstable, while thin black rings are radially stable. This suggest that the vacuum black rings with a single spin described in this paper can be stable only if the GL instability switches off for thin black rings with small enough j, a possibility that deserves further study. Vacuum black rings with two spins (yet to be constructed) presumably suffer also from superradiant instabilities peculiar to black objects with a rotating two-sphere [70]. The dipole charge in black rings may help stabilize them against some perturbations, like GL modes, in particular near the extremal limit.

[15] Even for these, the study of gravitational perturbations may only be tractable analytically in special cases with enhanced symmetry arising when some angular momenta coincide [64].

8.2 Generalizations

The number of degrees of freedom of gravity increases with the spacetime dimension D, so it is natural to expect more complex dynamics as D grows. In $D < 4$ the dynamics is so constrained that gravity has no propagating degrees of freedom. In $D = 4$ gravity does propagate, but it is still highly constrained, as the black hole uniqueness theorems illustrate. The discovery of black rings shows that $D = 5$ allows for more freedom but the dynamics is still amenable to detailed study. Gravity in $D > 5$ remains largely unexplored, but there are indications that black holes (even spherical ones) possess qualitatively new features [48].

It is natural to wonder whether black rings of horizon topology $S^1 \times S^{D-3}$ exist for $D > 5$. What about other topologies in $D > 5$, *e.g.,* $S^1 \times S^1 \times S^2$, $S^3 \times S^3$ etc? These possibilities are all consistent with the higher-dimensional topology theorem of [71].[16]

Heuristically, the balance of forces in thin black rings happens between centrifugal repulsion and tension [67, 68], which are independent of the number of dimensions since both forces are confined to the plane of the ring. The dimension-dependent gravitational force decays faster with the distance, so it plays a role only in the equilibrium of rings at small radii. This suggests that thin black rings should also exist in $D > 5$, and, like in $D = 5$, be unstable against radial perturbations. Observe that, since there is no bound on the angular momentum of MP black holes with a single spin in $D > 5$ [4], the existence of these rotating black rings *would* automatically imply the violation of black hole uniqueness.

Another possibility is that there might exist black rings with less symmetry than any known solution. It has been proved that a higher-dimensional stationary rotating black hole must admit a rotational symmetry [72]. However, known black rings (indeed, all known $D \geq 5$ black hole solutions) have multiple rotational symmetries. This has led to the suggestion that there may exist black hole solutions with less symmetry than the known solutions [21]. An example would be a vacuum black ring solution with the same charges as the one of [1] but lacking the "accidental" rotational symmetry $\partial/\partial\phi$ on the S^2.

8.3 From microscopics to macroscopics

Typically, the approach to black hole entropy calculations has been to start from a black hole solution, obtain its entropy from the area of the event horizon, and then try to reproduce this result statistically from a microscopic theory. However, one could just as well work backwards by constructing a microscopic model that gives rise to a macroscopic entropy and thereby predicting the existence of an associated black hole solution. For example, a formula for the microscopic entropy of the yet-to-be-found 9-parameter black ring solution discussed

[16] It will be up to their discoverers to find a good name for these black holes.

above was proposed in [52] based on U-duality of the IR theory[17]. It will be interesting to see whether this kind of approach can be pushed further, for example to predict properties of new black holes in $D > 5$. In general, such predictions are not much help in finding black hole solutions. However, special cases in which some supersymmetry is preserved may be more tractable, as we have seen is the case in $D = 5$.

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